BASIN BOUNDARIES OF PLANE MODEL OF IMPACTING COIN

P. Frantík, M. Štafa [1]

Summary: *The paper is focused on numerical plane model of a rigid coin with rectangular cross-section impacting to an elastic foundation.*

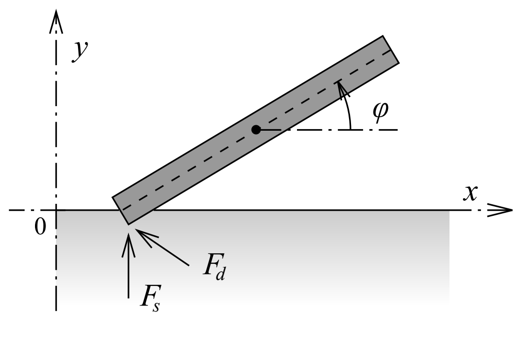
1. Introduction

Behavior of a rigid object repeatedly impacting to a surface of an elastic body is strongly dependent on properties of the contact and could present chaotic motion, e.g. see Peterka and Tondl (2002). There are several phenomena influencing direction and size of contact forces between the object and the body: Resistance of the surface against penetration of a shape, energy dissipation due to elastic properties of the body and friction between contact surfaces.

In this paper we will concentrate on a problem, which can be interpreted as a typical cylindrical coin restrained to move only in a plane with three degrees of freedom such as the cut by the plane will be a rectangle.

2. Model

The coin of diameter $d = 2$ cm, thickness $t = 2$ mm, weight $m = 3.14$ g, moment of inertia $I = 7.06 \cdot 10^{-8}$ kg m^2, is assumed as a rigid rectangle moving in the plane as shown on fig. 1. Its state is represented by three coordinates x, y, φ and their velocities v_x, v_y, v_φ.



Obrázek 1: Model of a coin

The coin is moving in gravity of size $g = 9.81$ m s^{-2}. The contact between the coin and the body is reduced on the penetration of corner points. Each corner point is loaded by contact force F composed of two forces: statical component F_s and damping component F_d acting only if

[1] Ing. Petr Frantík, Ph.D., Ing. Michal Štafa, Institute of Structural Mechanics, Faculty of Civil Engineering, Brno University of Technology, Veveří 331/95, 602 00 Brno, e-mail: kitnarf@centrum.cz

the corner is under the body surface. The statical component is considered simply as reaction of linear Winkler foundation:

$$F_{si}(y_i) = -k\, y_i, \tag{1}$$

where $k = 10000$ N/m is the stiffness of the contact and y_i is vertical coordinate of the corner i (the surface has zero y coordinate). Damping force is assumed as linear viscous damping loading the corner in the opposite direction of its velocity:

$$\begin{aligned} F_{dxi}(v_{xi}) &= -c\, v_{xi}, \\ F_{dyi}(v_{yi}) &= -c\, v_{yi}, \end{aligned} \tag{2}$$

where $c = 0.5$ Ns/m is the viscous damping coefficient and v_{xi}, v_{yi} are velocity components of the corner i. Corner coordinates are calculated in the following way:

$$\begin{aligned} x_1 &= x + d_{x1}, \;\; d_{x1} = r_d \cos\varphi - r_t \sin\varphi, \\ x_2 &= x + d_{x2}, \;\; d_{x2} = -r_d \cos\varphi - r_t \sin\varphi, \\ x_3 &= x + d_{x3}, \;\; d_{x3} = -r_d \cos\varphi + r_t \sin\varphi, \\ x_4 &= x + d_{x4}, \;\; d_{x4} = r_d \cos\varphi + r_t \sin\varphi, \\ \\ y_1 &= y + d_{y1}, \;\; d_{y1} = r_d \sin\varphi - r_t \cos\varphi, \\ y_2 &= y + d_{y2}, \;\; d_{y2} = -r_d \sin\varphi - r_t \cos\varphi, \\ y_3 &= y + d_{y3}, \;\; d_{y3} = -r_d \sin\varphi + r_t \cos\varphi, \\ y_4 &= y + d_{y4}, \;\; d_{y4} = r_d \sin\varphi + r_t \cos\varphi, \end{aligned} \tag{3}$$

where x_i is the horizontal coordinate of the corner i and for position radiuses apply $r_d = d/2$, $r_t = t/2$. Velocity components v_x, v_y of a corner are calculated similarly:

$$\begin{aligned} v_{x1} &= v_x - v_c(\sin\varphi \cos\alpha + \cos\varphi \sin\alpha), \\ v_{x2} &= v_x + v_c(\sin\varphi \cos\alpha - \cos\varphi \sin\alpha), \\ v_{x3} &= v_x + v_c(\sin\varphi \cos\alpha + \cos\varphi \sin\alpha), \\ v_{x4} &= v_x - v_c(\sin\varphi \cos\alpha - \cos\varphi \sin\alpha), \\ \\ v_{y1} &= v_y + v_c(\sin\varphi \sin\alpha - \cos\varphi \cos\alpha), \\ v_{y2} &= v_y - v_c(\sin\varphi \sin\alpha + \cos\varphi \cos\alpha), \\ v_{y3} &= v_y - v_c(\sin\varphi \sin\alpha - \cos\varphi \cos\alpha), \\ v_{y4} &= v_y + v_c(\sin\varphi \sin\alpha + \cos\varphi \cos\alpha), \end{aligned} \tag{4}$$

where v_c is the relative circumferencial velocity of the corners, α is the angle between coin symmetry axis and the diagonal for which following applies:

$$\begin{aligned} v_c &= r\, v_\varphi, \\ \sin\alpha &= r_t/r, \\ \cos\alpha &= r_d/r, \\ r &= \sqrt{r_d^2 + r_t^2}. \end{aligned} \tag{5}$$

After calculations of contact forces acting on the corners, the resultant acting on the coin centroid is calculated:

$$
\begin{aligned}
F_x &= \sum_{i=1}^{4} F_{dxi}, \\
F_y &= -mg + \sum_{i=1}^{4} (F_{dyi} + F_{si}), \\
M &= \sum_{i=1}^{4} \left(-F_{dxi}\, d_{yi} + (F_{dyi} + F_{si})\, d_{xi}\right),
\end{aligned}
\tag{6}
$$

3. Stability of static states

Before analysis of dynamical behavior we will look on static states and quantification of their stability. For the rigid rectangle and rigid body the function of potential energy V takes form of an ellipse originating from circular motion of the centroid around a corner:

$$V(\varphi) = mg\,(r_t \cos\varphi + r_d \sin\varphi),\;\; \varphi \in \langle 0, \pi/2 \rangle. \tag{7}$$

Its minimal and maximal values are following:

$$
\begin{aligned}
&V_{max} = V(\varphi_{cr}) = mg\,r,\;\; \varphi_{cr} = \arctan(r_d/r_t), \\
&V_{min1} = V(0) = mg\,r_t, \\
&V_{min2} = V(\pi/2) = mg\,r_d.
\end{aligned}
\tag{8}
$$

For the loss of the stability of each of two statical states it is neccessary to add to the coin at least $V_{max} - V_{min1}$ Joules or $V_{max} - V_{min2}$ Joules respectively. The ratio between these values is given by relation:

$$\frac{r - r_t}{r - r_d} \approx 181.4 \tag{9}$$

It means that the coin laying on the surface of a rigid body is almost 200× more stable than the standing coin.

If the body is elastic the situation is much more interesting. At first it applies that all three static states exists for the same angle φ as for rigid case. Than their potential energy can be derived to the form:

$$
\begin{aligned}
&V_{max} = V(\varphi_{cr}) = mg\,(r - \tfrac{mg}{k}), \\
&V_{min1} = V(0) = mg\,(r_t - \tfrac{mg}{2k}), \\
&V_{min2} = V(\pi/2) = mg\,(r_d - \tfrac{mg}{2k}),
\end{aligned}
\tag{10}
$$

whereas single corner contact is considered for the critical angle φ_{cr}. This condition is satisfied for the contact stiffness $k \geq k_{min}$, for which applies:

$$k_{min} = \frac{mg\,r}{2r_t^2} \approx 154.9\,\mathrm{N/m}. \tag{11}$$

As for rigid case we can derive ratio between values of energetic potential needed to loss of stability of static states:

$$\frac{r - r_t - mg/2k}{r - r_d - mg/2k} \approx 187.2 \tag{12}$$

Note that deviation between rigid and elastic solution for chosen stiffness is only about 3%. This relation does not have real value for the denominator equal to zero. It can be signed as critical contact stiffness k_{cr}. The standing coin could be at static state only if the stiffness k is higher than this critical value (a bifurcation point):

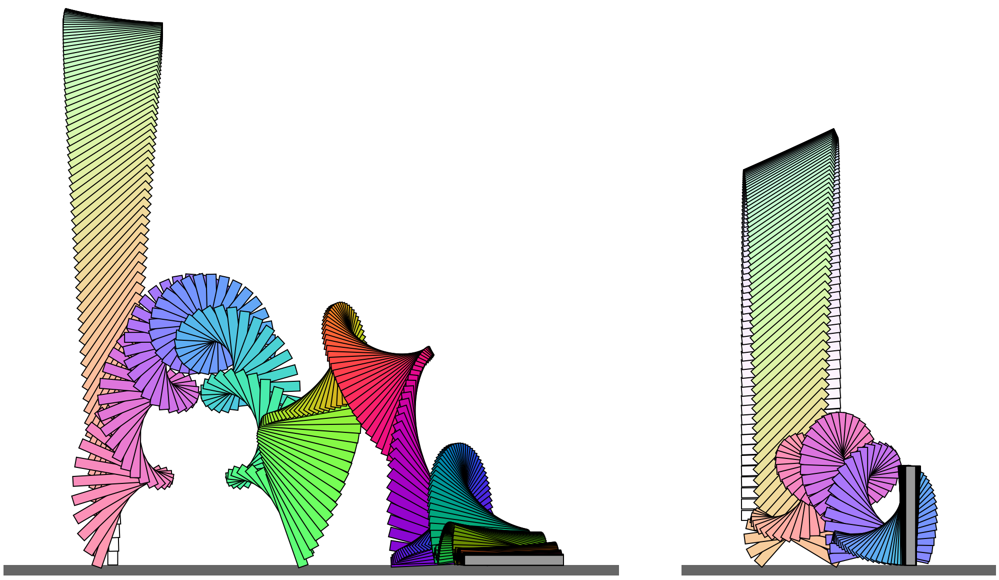
$$k_{cr} = \frac{mg}{2(r_d - r)} \approx 308.96\,\mathrm{N/m}. \tag{13}$$

4. Dynamical simulation

Equations of motion of the coin are given by classical approach in the following form as dynamical system:

$$\begin{aligned} &\frac{\mathrm{d}v_x}{\mathrm{d}t} = \frac{F_x}{m}, \quad \frac{\mathrm{d}v_y}{\mathrm{d}t} = \frac{F_y}{m}, \quad \frac{\mathrm{d}v_\varphi}{\mathrm{d}t} = \frac{M}{I}, \\ &\frac{\mathrm{d}x}{\mathrm{d}t} = v_x, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = v_y, \quad \frac{\mathrm{d}\varphi}{\mathrm{d}t} = v_\varphi, \end{aligned} \tag{14}$$

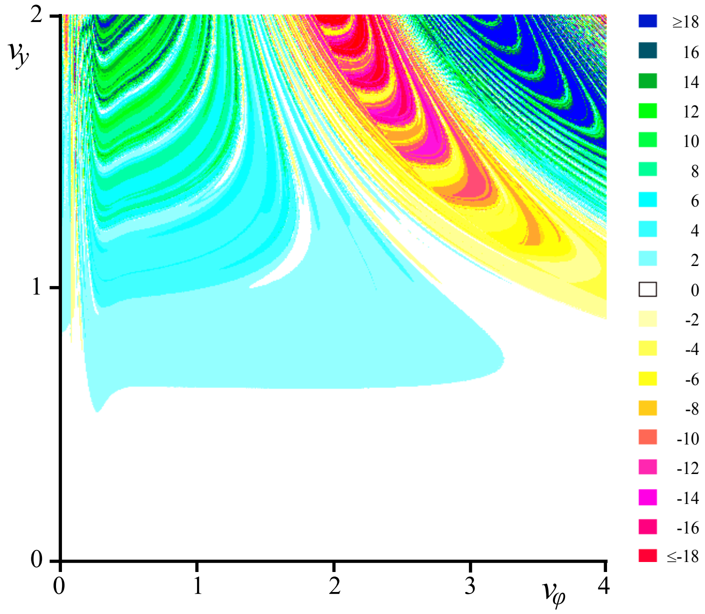
where t is time. This system is solved numerically by the Symplectic Euler method, see e.g. Hairer (2010), with step $h = 10^{-6}$ s. Each simulation starts with specific initial conditions and is terminated after two seconds of the motion of the coin. The values of initial conditions are taken from ranges which ensures that after two seconds a stable state will be achieved. On fig. 2 two simulation results are shown. One for typical solution where the coin lay at the end (on the left) and second where the coin stands[2] (on the right).



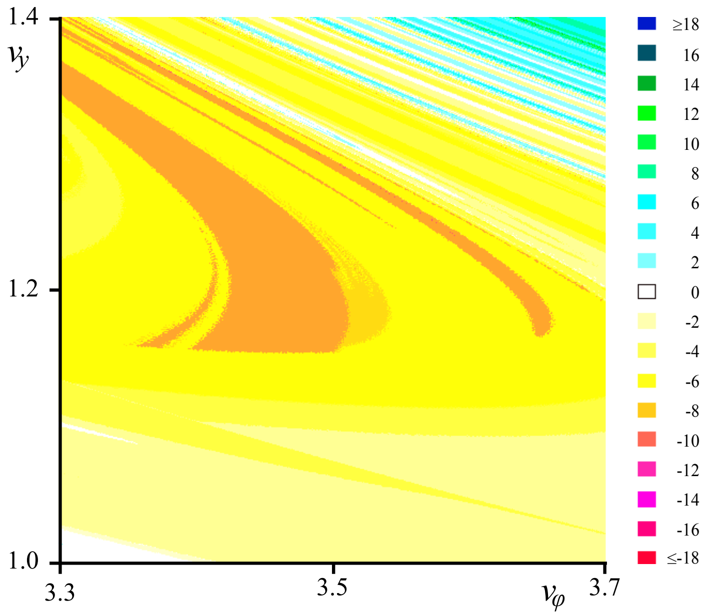
Obrázek 2: Phases of two simulations; the colors are darker as time runs out

[2] $y_0 = r$, $v_{y0} = 1.197$ m/s, $v_{\varphi 0} = 3.508$ rad/s and the others are zero

The coin has six initial conditions: $x_0, y_0, \varphi_0, v_{x0}, v_{y0}, v_{\varphi 0}$. For the basin boundaries the initial horizontal position x_0 is obviously irrelevant. Therefore only five initial conditions have an influence on basin boundaries. On the current processors we need approximately one half of second to calculate one simulation. If we use raw sampling with 128 samples of each of five condition then we need to calculate 128^5 simulations ($34\,360$ millions) for the whole initial conditions space. It means rawly 544 years of sequential calculation. We simplified the problem to variation of only two initial conditions $v_{y0}, v_{\varphi 0}$. Other conditions are set as follows: $x_0 = 0$ m, $y_0 = r$, $\varphi_0 = 0$ rad, $v_{x0} = 0$ m/s. On the fig. 3-5 calculated basin boundaries are shown with 512 samples on each axis.



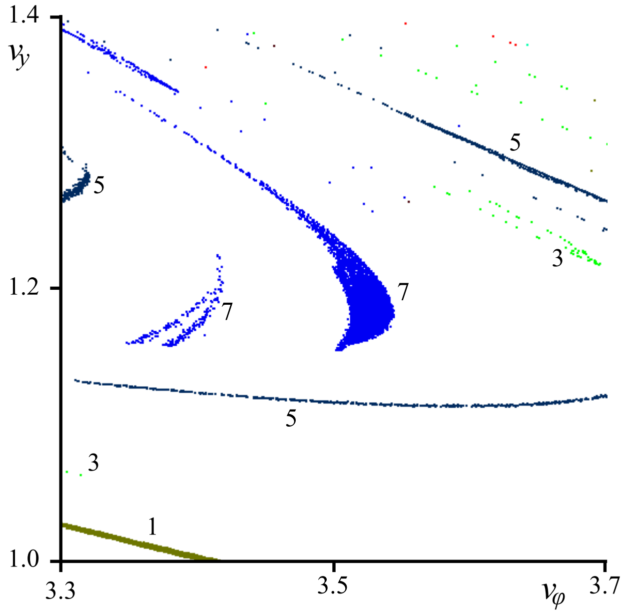
Obrázek 3: Basin boundaries showing number of $\pi/2$ rotations



Obrázek 4: Detail of an interesting part of previous figure

Colors in the figures represents integer values $n = 2\varphi/\pi$ which means number of $\pi/2$ rotations passed until the final state was reached. Yellow-red colors sign clockwise rotation, green-blue colors counterclockwise rotation. Even numbers means that coin lay, odd numbers are valid for standing coin.

On fig. 3 are 635 particular values of initial conditions from $512^2 = 262\,144$ samples. It means that probability of achieving the standing coin is approximately 0.24% form chosen subset of initial condition space. Fig. 5 enlarges an expressive area from which the coin stands. Initial conditions from this area leads to seven $\pi/2$ rotations as shown on fig. 2 on right side.



Obrázek 5: Basins where the coin stands

5. Conclusions

The chosen model of a coin impacting to an elastic body was described in detail. The coin was restrained to move in a plane and dependency of its final state on the initial conditions was studied. Static states was also analysed and critical values for the elastic contact were determined.

6. Acknowledgment

This outcome has been achieved with the financial support of the project No. FAST-S-12-34.

7. Reference

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